

# Star-product Quantization in Second-class Constraint Systems

M. Nakamura\*

*Department of Service and Business, Hamamatsu University, Miyakoda-cho 1230,  
Kita-ku, Hamamatsu-shi, Shizuoka 431-2102, Japan*

## Abstract

The quantization of the second-class constraint systems is discussed within the projection operator method(POM) of constraint systems. Using the nonlocal representation of the projection *hyper* operators, the new star products are proposed, which are composed of the constraint *hyper* operators. Then, the projected operator-algebra of the quantized constraint systems is given with these star-products, and it is shown that the commutators and symmetrized products among the projected operators contain the quantum corrections due to the noncommutativity among operators in the product of the projected operators.

PACS numbers: 11.10.Ef, 03.65.-W

## 1 Introduction

The problem of the quantization of constraint systems has been extensively investigated as one of the most fundamental problems in gauge theory[1]. There exist two standard approaches to quantize constraint systems. The first approach, which we shall call the approach I, is to impose the constraints first and then to quantize on the reduced phase space[2]. The second, which we shall call the approach II, is inversely first to quantize on the unconstrained phase space, and then to impose the constraints as the operator-equations[3].

There often occur the situations where the two are not equivalent. In the case of the first-class constraints, this problem has been extensively investigated until now, and it has been shown that the second approach involves the contributions which can never appear in the first one[4].

---

\*E-mail:mnakamur@hamamatsu-u.ac.jp

In the case of second-class constraints, the quantization of the constraint systems in the canonical formalism has usually been accomplished by using the generalized Hamiltonian formalism with the Dirac bracket [2], which is in the approach I. There have been proposed many formalism within approach II; the operator formalism developed by Batalin and Fradkin[3], the projection operator method[5, 6] and the algebraic approach proposed by Ohnuki and Kitakado[7], etc. Using the projection operator method[5] (POM), then, we have shown that the operator-algebra in the approach (II) contain the quantum corrections caused by the noncommutativity in the re-ordering of constraint-operators in the products of operators, which can never be obtained by the approach (I). Alternative formalism of quantization is the Weyl-Wigner-Moyal(WWM) quantization[8, 9], and has been investigated for the constraint systems in Refs.[10, 11]. In the WWM, the star-product are defined in the following form[9, 12]:

$$\star = \exp \left( \frac{i\hbar}{2} \theta^{\mu\nu} \overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu \right), \quad (1.1)$$

where  $\partial_\mu = \partial/\partial z_\mu$  with the canonical coordinates and momenta  $z_\mu = (q^i, p_i; i = 1, \dots, N)$ , and  $\theta^{\mu\nu}$  is the symplectic matrix.

By using the nonlocal representation of the operations of *hyper*-operators  $\hat{\xi}$  and  $\hat{\xi}'$  on the operators  $X$  and  $Y$ ,

$$(\hat{\xi}X)(\hat{\xi}'Y) = \hat{\xi}(k)\hat{\xi}'(l)X(k)Y(l)|_{k=l}, \quad (1.2)$$

where  $k = z_\mu^{(k)}$  is a set of the operators  $(q_{(k)}^i, p_i^{(k)}; i = 1, \dots, N)$ , we shall propose the new star-products, which are composed with the constraint *hyper*-operators and have the same structures as (1.1). Then, the projected operator-algebra formulated in POM is reformulated in terms of these new star-products and the reformulated algebra is shown to be the extension of the Moyal star-products to the operator formalism in the constraint systems.

The present paper is organized as follows. In sect. 2, we review the POM in terms of the symplectic representation. In sect. 3, we introduce the nonlocal representation for the operations of *hyper*-operators in the form of (1.2) and the new star-products are proposed. By using these star-products, we present the general formulas of the commutators and symmetrized products among the projected operators in the second-class constraint system. It is shown that these commutators and symmetrized products contain the quantum correction terms due to the noncommutativity of operators. In sect. 4, some concluding remarks are given.

## 2 Projection Operator Method of Constraint System

We shall treat the constraint systems with the supersymmetry. So, we will adopt the supercommutator defined as

$$[A, B] = AB - (-1)^{\epsilon(A)\epsilon(B)} BA \quad (2.1)$$

and the supersymmetrized product

$$\{A, B\} = \frac{1}{2}(AB + (-1)^{\epsilon(A)\epsilon(B)} BA) \quad (2.2)$$

for operators  $A$  and  $B$ , where  $\epsilon(A)$  is the Grassmann parity of  $A$ . Let  $\hat{A}^{(+)}$ ,  $\hat{A}^{(-)}$  be the *hyper*-operators defined as follows: For any operator  $O$ ,

$$\hat{A}^{(+)}O = \{A, O\}, \quad \hat{A}^{(-)}O = \frac{1}{i\hbar}[A, O]. \quad (2.3)$$

Then, they obey the *hyper*-commutator algebra

$$\begin{aligned} [\hat{A}^{(+)}, \hat{B}^{(+)}] &= \frac{i\hbar}{4}\hat{C}_{AB}^{(-)}, \\ [\hat{A}^{(-)}, \hat{B}^{(-)}] &= \frac{1}{i\hbar}\hat{C}_{AB}^{(-)}, \\ [\hat{A}^{(\pm)}, \hat{B}^{(\mp)}] &= \frac{1}{i\hbar}\hat{C}_{AB}^{(+)}, \end{aligned} \quad (2.4)$$

where  $C_{AB} = [A, B]$ .

### 2.1 Second-class constraint system

Let  $(\mathcal{C}, H(\mathcal{C}), T_\alpha(\mathcal{C}))$  be the initial unconstraint quantum system, where  $\mathcal{C} = \{(q^i, p_i); i = 1, \dots, N\}$  is a set of canonically conjugate operators (CCS), which satisfy the canonical commutation relations (CCR)

$$[q^i, p_j] = i\hbar\delta_j^i, \quad [q^i, q^j] = [p_i, p_j] = 0, \quad (2.5)$$

$H(\mathcal{C})$ , the Hamiltonian of the initial unconstraint system and  $T_\alpha(\mathcal{C})$  ( $\alpha = 1, \dots, 2M < 2N$ ), the constraint-operators corresponding to the second-class constraints  $T_\alpha = 0$ . Starting with  $(\mathcal{C}, H(\mathcal{C}), T_\alpha(\mathcal{C}))$ , then, we construct the constraint quantum system

$(\mathcal{C}^*, H^*(\mathcal{C}^*), T_\alpha(\mathcal{C}^*))$ , where  $\mathcal{C}^*$  is the set of  $N - M$  canonically conjugate pairs satisfying

$$T_\alpha(\mathcal{C}^*) = 0 \quad (\alpha = 1, \dots, 2M). \quad (2.6)$$

From the analogues of the treatment with the Darboux's theorem in the classical systems, we can, in principle, construct the canonically conjugate set in terms of  $T_\alpha(\mathcal{C})$ , which we call the associated canonically conjugate set (ACCS)[5].

Let  $\{(\xi^a, \pi_a) | \epsilon(\xi^a) = \epsilon(\pi_a) = s, a = 1, \dots, M\}$  be the ACCS constructed with  $T_\alpha$ . In order to collectively represent the ACCS, we introduce the symplectic form  $Z_\alpha$  as follows:

$$Z_\alpha = \begin{cases} \xi^a & (\alpha = a) \\ \pi_a & (\alpha = a + n) \end{cases} \quad (\alpha = 1, \dots, 2n \quad ; \quad a = 1, \dots, n), \quad (2.7)$$

which obeys the commutation relation

$$[Z_\alpha, Z_\beta] = i\hbar(-(-)^s)J_{\alpha\beta} = i\hbar J^{\alpha\beta}, \quad (2.8)$$

where

$$J^{\alpha\beta} = \begin{pmatrix} O & I \\ -(-)^s I & O \end{pmatrix}_{\alpha\beta} \quad \text{with} \begin{cases} I : n \times n \text{ unit matrix} \\ O : n \times n \text{ zero matrix} \end{cases} \quad (2.9a)$$

is the supersymmetric symplectic matrix and  $J_{\alpha\beta}$ , defined as

$$J_{\alpha\beta} = \begin{pmatrix} O & -(-)^s I \\ I & O \end{pmatrix}_{\alpha\beta}, \quad (2.9b)$$

is the inverse of  $J^{\alpha\beta}$  with

$$J^{\alpha\beta} J_{\beta\gamma} = \delta_\gamma^\alpha, \quad J_{\alpha\beta} J^{\beta\gamma} = \delta_\alpha^\gamma. \quad (2.9c)$$

From (2.4) and (2.8), the symplectic *hyper*-operators obey the *hyper*-commutation relations

$$\begin{aligned} [\hat{Z}_\alpha^{(\pm)}, \hat{Z}_\beta^{(\pm)}] &= 0, \\ [\hat{Z}_\alpha^{(\pm)}, \hat{Z}_\beta^{(\mp)}] &= [\hat{Z}_\alpha^{(\mp)}, \hat{Z}_\beta^{(\pm)}] = J^{\alpha\beta}. \end{aligned} \quad (2.10)$$

## 2.2 Projection operator $\hat{\mathcal{P}}$ and projected system

Following Ref.[5], we define the *hyper*-operator  $\hat{\mathcal{P}}$  as follows:

$$\begin{aligned} \hat{\mathcal{P}} &= \sum_{n=0}^{\infty} \frac{(-)^{ns}}{n!} J^{\alpha_1\beta_1} \dots J^{\alpha_n\beta_n} \hat{Z}_{\alpha_1}^{(+)} \dots \hat{Z}_{\alpha_n}^{(+)} \hat{Z}_{\beta_n}^{(-)} \dots \hat{Z}_{\beta_1}^{(-)} \\ &= \exp \left[ (-1)^s \hat{Z}_\alpha^{(+)} \frac{\partial}{\partial \varphi_\alpha} \right] \exp [J^{\alpha\beta} \varphi_\alpha \hat{Z}_\beta^{(-)}] |_{\phi=0}, \end{aligned} \quad (2.11)$$

which satisfies the following algebraic properties[5]:

$$\begin{aligned}
\hat{\mathcal{P}} \cdot \hat{\mathcal{P}} &= \hat{\mathcal{P}}, \\
\sum_{n=0}^{\infty} \frac{(-1)^{(s+1)n}}{n!} J^{\alpha_1 \beta_1} \dots J^{\alpha_n \beta_n} \hat{Z}_{\alpha_1}^{(+)} \dots \hat{Z}_{\alpha_n}^{(+)} \hat{\mathcal{P}} \hat{Z}_{\beta_n}^{(-)} \dots \hat{Z}_{\beta_1}^{(-)} \\
&= \exp \left[ -(-1)^s \hat{Z}_{\alpha}^{(+)} \frac{\partial}{\partial \varphi_{\alpha}} \right] \hat{\mathcal{P}} \exp [J^{\alpha \beta} \varphi_{\alpha} \hat{Z}_{\beta}^{(-)}] |_{\varphi=0} \\
&= \exp \left[ (-1)^s \hat{Z}_{\alpha}^{(+)} \frac{\partial}{\partial \varphi_{\alpha}} \right] \hat{\mathcal{P}} \exp [-J^{\alpha \beta} \varphi_{\alpha} \hat{Z}_{\beta}^{(-)}] |_{\varphi=0} = 1,
\end{aligned} \tag{2.12}$$

$$\hat{\mathcal{P}} Z_{\alpha} = 0, \quad \hat{\mathcal{P}} \hat{Z}_{\alpha}^{(+)} = \hat{Z}_{\alpha}^{(-)} \hat{\mathcal{P}} = 0. \tag{2.13}$$

Then, the available formulas of the projections of commutators and symmetrized products including  $\hat{Z}^{(+)}$  are obtained, which we shall present in Appendix A.

Through the operation of  $\hat{\mathcal{P}}$ , now, the initial unconstraint system  $(\mathcal{C}, H(\mathcal{C}), T_{\alpha}(\mathcal{C}))$  is projected on to the constraint system  $(\mathcal{C}^*, H^*(\mathcal{C}^*), T_{\alpha}(\mathcal{C}^*))$  as follows:

$$(\mathcal{C}, H(\mathcal{C}), T_{\alpha}(\mathcal{C})) \longmapsto (\mathcal{C}^*, H^*(\mathcal{C}^*), T_{\alpha}(\mathcal{C}^*)), \tag{2.14}$$

and the operator  $O \in \mathcal{C}$  is transformed to  $O^* = \hat{\mathcal{P}} O \in \mathcal{C}^*$ , where

$$\begin{aligned}
\mathcal{C}^* &= \hat{\mathcal{P}} \mathcal{C} = \{(q^{*1}, p_1^*), \dots, (q^{*N^*}, p_{N^*}^*)\}, \quad (N^* = N - M), \\
H^*(\mathcal{C}^*) &= \hat{\mathcal{P}} H(\mathcal{C}) = H^*(\hat{\mathcal{P}} \mathcal{C}), \\
T_{\alpha}(\mathcal{C}^*) &= \hat{\mathcal{P}} T_{\alpha}(\mathcal{C}) = 0.
\end{aligned} \tag{2.15}$$

The condition  $T_{\alpha}(\mathcal{C}^*) = 0$  is called as the pojection condition[5].

### 2.3 Commutator and Symmetrized product formulas of projected operators

By using the algebraic properties (2.12), (2.13) and (A.1)-(A.4), we obtain the following formulas in the constrained system  $(\mathcal{C}^*, H^*(\mathcal{C}^*), T_{\alpha}(\mathcal{C}^*))$ : for any operator  $X$  and  $Y$ ,

$$[\hat{\mathcal{P}} X, \hat{\mathcal{P}} Y] = \sum_{n=0}^{\infty} (-1)^{ns} C^{(2n)}(X, Y) + 2(-1)^{\varepsilon_X s} \sum_{n=0}^{\infty} (-1)^{ns+s} S^{(2n+1)}(X, Y), \tag{2.16a}$$

$$\{\hat{\mathcal{P}} X, \hat{\mathcal{P}} Y\} = \sum_{n=0}^{\infty} (-1)^{ns} S^{(2n)}(X, Y) + \frac{1}{2}(-1)^{\varepsilon_X s} \sum_{n=0}^{\infty} (-1)^{ns+s} C^{(2n+1)}(X, Y), \tag{2.16b}$$

$$\hat{\mathcal{P}}[X, Y] = \sum_{n=0}^{\infty} (-1)^{ns} C_{\hat{\mathcal{P}}}^{(2n)}(X, Y) - 2(-1)^{\varepsilon_X s} \sum_{n=0}^{\infty} (-1)^{ns+s} S_{\hat{\mathcal{P}}}^{(2n+1)}(X, Y), \quad (2.17a)$$

$$\hat{\mathcal{P}}\{X, Y\} = \sum_{n=0}^{\infty} (-1)^{ns} S_{\hat{\mathcal{P}}}^{(2n)}(X, Y) - \frac{1}{2}(-1)^{\varepsilon_X s} \sum_{n=0}^{\infty} (-1)^{ns+s} C_{\hat{\mathcal{P}}}^{(2n+1)}(X, Y), \quad (2.17b)$$

where

$$C^{(n)}(X, Y) = \frac{1}{n!} \left( \frac{\hbar}{2i} \right)^n J^{\alpha_1 \beta_1} \dots J^{\alpha_n \beta_n} \hat{\mathcal{P}}[\hat{Z}_{\alpha_n}^{(-)} \dots \hat{Z}_{\alpha_1}^{(-)} X, \hat{Z}_{\beta_n}^{(-)} \dots \hat{Z}_{\beta_1}^{(-)} Y], \quad (2.18)$$

$$S^{(n)}(X, Y) = \frac{1}{n!} \left( \frac{\hbar}{2i} \right)^n J^{\alpha_1 \beta_1} \dots J^{\alpha_n \beta_n} \hat{\mathcal{P}}\{\hat{Z}_{\alpha_n}^{(-)} \dots \hat{Z}_{\alpha_1}^{(-)} X, \hat{Z}_{\beta_n}^{(-)} \dots \hat{Z}_{\beta_1}^{(-)} Y\},$$

and

$$C_{\hat{\mathcal{P}}}^{(n)}(X, Y) = \frac{1}{n!} \left( \frac{\hbar}{2i} \right)^n J^{\alpha_1 \beta_1} \dots J^{\alpha_n \beta_n} [\hat{\mathcal{P}} \hat{Z}_{\alpha_n}^{(-)} \dots \hat{Z}_{\alpha_1}^{(-)} X, \hat{\mathcal{P}} \hat{Z}_{\beta_n}^{(-)} \dots \hat{Z}_{\beta_1}^{(-)} Y],$$

$$S_{\hat{\mathcal{P}}}^{(n)}(X, Y) = \frac{1}{n!} \left( \frac{\hbar}{2i} \right)^n J^{\alpha_1 \beta_1} \dots J^{\alpha_n \beta_n} \{\hat{\mathcal{P}} \hat{Z}_{\alpha_n}^{(-)} \dots \hat{Z}_{\alpha_1}^{(-)} X, \hat{\mathcal{P}} \hat{Z}_{\beta_n}^{(-)} \dots \hat{Z}_{\beta_1}^{(-)} Y\}. \quad (2.19)$$

### 3 Star-product Representaion of Projected operator Algebra

#### 3.1 Nonlocal representation of Projected operator Algebra

Let the operator  $X$ , the *hyper*-operators  $\hat{Z}_{\alpha}^{(-)}$  and  $\hat{\mathcal{P}}$  in the CCS  $\mathcal{C}_{\eta}$  be  $X(\eta)$ ,  $\hat{Z}_{\alpha}^{(-)}(\eta)$  and  $\hat{\mathcal{P}}(\eta)$ , respectively. Then, the product of  $(\hat{Z}_{\alpha}^{(-)} X)(\hat{Z}_{\beta}^{(-)} Y)$  is rewritten as

$$(\hat{Z}_{\alpha}^{(-)} X)(\hat{Z}_{\beta}^{(-)} Y) = \left( (-1)^{\varepsilon_X s} \hat{Z}_{\alpha}^{(-)}(\eta) \hat{Z}_{\beta}^{(-)}(\zeta) X(\eta) Y(\zeta) \right) \Big|_{\eta=\zeta} \quad (3.1)$$

From (3.1), the products of such operators as  $\hat{Z}_{\alpha_n}^{(-)} \dots \hat{Z}_{\alpha_1}^{(-)} X$  are rewritten as follows:

$$\begin{aligned} & (\hat{Z}_{\alpha_{2n}}^{(-)} \dots \hat{Z}_{\alpha_1}^{(-)} X)(\hat{Z}_{\beta_{2n}}^{(-)} \dots \hat{Z}_{\beta_1}^{(-)} Y) \\ &= (-1)^{ns} \left( \hat{Z}_{\alpha_{2n}}^{(-)}(\eta) \hat{Z}_{\beta_{2n}}^{(-)}(\zeta) \dots \hat{Z}_{\alpha_1}^{(-)}(\eta) \hat{Z}_{\beta_1}^{(-)}(\zeta) X(\eta) Y(\zeta) \right) \Big|_{\eta=\zeta}, \end{aligned} \quad (3.2a)$$

$$\begin{aligned} & (\hat{Z}_{\alpha_{2n+1}}^{(-)} \dots \hat{Z}_{\alpha_1}^{(-)} X)(\hat{Z}_{\beta_{2n+1}}^{(-)} \dots \hat{Z}_{\beta_1}^{(-)} Y) \\ &= (-1)^{ns+\varepsilon_X s} \left( \hat{Z}_{\alpha_{2n+1}}^{(-)}(\eta) \hat{Z}_{\beta_{2n+1}}^{(-)}(\zeta) \dots \hat{Z}_{\alpha_1}^{(-)}(\eta) \hat{Z}_{\beta_1}^{(-)}(\zeta) X(\eta) Y(\zeta) \right) \Big|_{\eta=\zeta}, \end{aligned} \quad (3.2b)$$

and then, the product of the projected operators becomes as

$$(\hat{\mathcal{P}}X)(\hat{\mathcal{P}}Y) = \left( \hat{\mathcal{P}}(\eta)\hat{\mathcal{P}}(\zeta)X(\eta)Y(\zeta) \right) \Big|_{\eta=\zeta}. \quad (3.2c)$$

By using the formulas (3.2a), (3.2b) and (3.2c), one obtains the following nonlocal representations of the commutators and the symmetrized products among the operators under the operation of  $\hat{\mathcal{P}}$ :

$$\begin{aligned} [\hat{\mathcal{P}}X, \hat{\mathcal{P}}Y] &= \hat{\mathcal{P}} \left( \cosh\left(\frac{\hbar}{2i}\hat{Z}_\alpha^{(-)}(\eta)J^{\alpha\beta}\hat{Z}_\beta^{(-)}(\zeta)\right)[X(\eta), Y(\zeta)] \right) \Big|_{\eta=\zeta} \\ &+ (-1)^s 2 \sinh\left(\frac{\hbar}{2i}\hat{Z}_\alpha^{(-)}(\eta)J^{\alpha\beta}\hat{Z}_\beta^{(-)}(\zeta)\right)\{X(\eta), Y(\zeta)\} \Big|_{\eta=\zeta}, \end{aligned} \quad (3.3a)$$

$$\begin{aligned} \{\hat{\mathcal{P}}X, \hat{\mathcal{P}}Y\} &= \hat{\mathcal{P}} \left( \cosh\left(\frac{\hbar}{2i}\hat{Z}_\alpha^{(-)}(\eta)J^{\alpha\beta}\hat{Z}_\beta^{(-)}(\zeta)\right)\{X(\eta), Y(\zeta)\} \right) \Big|_{\eta=\zeta} \\ &+ (-1)^s \frac{1}{2} \sinh\left(\frac{\hbar}{2i}\hat{Z}_\alpha^{(-)}(\eta)J^{\alpha\beta}\hat{Z}_\beta^{(-)}(\zeta)\right)[X(\eta), Y(\zeta)] \Big|_{\eta=\zeta}, \end{aligned} \quad (3.3b)$$

$$\begin{aligned} \hat{\mathcal{P}}[X, Y] &= \left( \hat{\mathcal{P}}(\eta)\hat{\mathcal{P}}(\zeta) \left( \cosh\left(\frac{\hbar}{2i}\hat{Z}_\alpha^{(-)}(\eta)J^{\alpha\beta}\hat{Z}_\beta^{(-)}(\zeta)\right)[X(\eta), Y(\zeta)] \right. \right. \\ &\left. \left. - (-1)^s 2 \sinh\left(\frac{\hbar}{2i}\hat{Z}_\alpha^{(-)}(\eta)J^{\alpha\beta}\hat{Z}_\beta^{(-)}(\zeta)\right)\{X(\eta), Y(\zeta)\} \right) \right) \Big|_{\eta=\zeta}, \end{aligned} \quad (3.4a)$$

$$\begin{aligned} \hat{\mathcal{P}}\{X, Y\} &= \left( \hat{\mathcal{P}}(\eta)\hat{\mathcal{P}}(\zeta) \left( \cosh\left(\frac{\hbar}{2i}\hat{Z}_\alpha^{(-)}(\eta)J^{\alpha\beta}\hat{Z}_\beta^{(-)}(\zeta)\right)\{X(\eta), Y(\zeta)\} \right. \right. \\ &\left. \left. - (-1)^s \frac{1}{2} \sinh\left(\frac{\hbar}{2i}\hat{Z}_\alpha^{(-)}(\eta)J^{\alpha\beta}\hat{Z}_\beta^{(-)}(\zeta)\right)[X(\eta), Y(\zeta)] \right) \right) \Big|_{\eta=\zeta}. \end{aligned} \quad (3.4b)$$

## 3.2 Star-product representation of Projected operator Algebra

### 3.2.1 Product of projected operators

By using the nonlocal representations (3.3a) and (3.3b), one obtains the Moyal product form of operator-equation for the product of projected operators  $(\hat{\mathcal{P}}X)(\hat{\mathcal{P}}Y)$ . Let  $\Theta^{\eta\zeta}$  be the bilinear form of  $\hat{Z}_\alpha^{(-)}(\eta)$ , which is defined by

$$\Theta^{\eta\zeta} = \hat{Z}_\alpha^{(-)}(\eta)J^{\alpha\beta}\hat{Z}_\beta^{(-)}(\zeta). \quad (3.5)$$

Then, the product  $(\hat{\mathcal{P}}X)(\hat{\mathcal{P}}Y)$  is rewritten in the following way:

$$\begin{aligned}
(\hat{\mathcal{P}}X)(\hat{\mathcal{P}}Y) &= \frac{1}{2}[\hat{\mathcal{P}}X, \hat{\mathcal{P}}Y] + \{\hat{\mathcal{P}}X, \hat{\mathcal{P}}Y\} \\
&= \hat{\mathcal{P}} \left( \left( \cosh\left(\frac{\hbar}{2i}\Theta^{\eta\zeta}\right) + (-1)^s \sinh\left(\frac{\hbar}{2i}\Theta^{\eta\zeta}\right) \right) X(\eta)Y(\zeta) \right) \Big|_{\eta=\zeta} \\
&= \hat{\mathcal{P}} \exp\left(\frac{\hbar}{2i}\Theta^{\eta\zeta}\right) \left( \frac{1}{2}(X(\eta)Y(\zeta) + X(\zeta)Y(\eta)) + \frac{(-1)^s}{2}(X(\eta)Y(\zeta) - X(\zeta)Y(\eta)) \right) \Big|_{\eta=\zeta}.
\end{aligned} \tag{3.6}$$

From (3.6), we obtain the following formulas of the commutator and the symmetrized product for the projected operators  $\hat{\mathcal{P}}X$  and  $\hat{\mathcal{P}}Y$ :

$$\begin{aligned}
[\hat{\mathcal{P}}X, \hat{\mathcal{P}}Y] &= (\hat{\mathcal{P}}X)(\hat{\mathcal{P}}Y) - (-1)^{\varepsilon_X \varepsilon_Y} (\hat{\mathcal{P}}Y)(\hat{\mathcal{P}}X) \\
&= (-1)^s \hat{\mathcal{P}} \exp\left(\frac{\hbar}{2i}\Theta^{\eta\zeta}\right) (X(\eta)Y(\zeta) - (-1)^{\varepsilon_X \varepsilon_Y} Y(\eta)X(\zeta)) \Big|_{\eta=\zeta},
\end{aligned} \tag{3.7a}$$

$$\begin{aligned}
\{\hat{\mathcal{P}}X, \hat{\mathcal{P}}Y\} &= \frac{1}{2}((\hat{\mathcal{P}}X)(\hat{\mathcal{P}}Y) + (-1)^{\varepsilon_X \varepsilon_Y} (\hat{\mathcal{P}}Y)(\hat{\mathcal{P}}X)) \\
&= \hat{\mathcal{P}} \exp\left(\frac{\hbar}{2i}\Theta^{\eta\zeta}\right) \frac{1}{2} (X(\eta)Y(\zeta) + (-1)^{\varepsilon_X \varepsilon_Y} Y(\eta)X(\zeta)) \Big|_{\eta=\zeta}.
\end{aligned} \tag{3.7b}$$

### 3.2.2 Projection of product of operators

By using the nonlocal representations of the projections of the commutator and symmetrized product, (3.4a) and (3.4b), one obtains the Moyal product form of operator-equation for the projection of the operator  $XY$  in the following way:

$$\begin{aligned}
\hat{\mathcal{P}}(XY) &= \frac{1}{2}(\hat{\mathcal{P}}[[X, Y] + 2\hat{\mathcal{P}}\{X, Y\}) \\
&= \left( \hat{\mathcal{P}}(\eta)\hat{\mathcal{P}}(\zeta) \left( \cosh\left(\frac{\hbar}{2i}\Theta^{\eta\zeta}\right) - (-1)^s \sinh\left(\frac{\hbar}{2i}\Theta^{\eta\zeta}\right) \right) X(\eta)Y(\zeta) \right) \Big|_{\eta=\zeta} \\
&= \left( \hat{\mathcal{P}}(\eta)\hat{\mathcal{P}}(\zeta) \exp\left(\frac{\hbar}{2i}\Theta^{\eta\zeta}\right) \right. \\
&\quad \times \left. \left( \frac{1}{2}(X(\eta)Y(\zeta) + X(\zeta)Y(\eta)) - \frac{(-1)^s}{2}(X(\eta)Y(\zeta) - X(\zeta)Y(\eta)) \right) \right) \Big|_{\eta=\zeta}.
\end{aligned} \tag{3.8}$$



Then, the projections of  $[X, Y]$  and  $\{X, Y\}$  are rewritten in following way:

$$\begin{aligned}\hat{\mathcal{P}}[X, Y] &= \hat{\mathcal{P}}(XY) - (-1)^{\varepsilon_X \varepsilon_Y} \hat{\mathcal{P}}(YX) \\ &= -(-1)^s \left( \hat{\mathcal{P}}(\eta) \hat{\mathcal{P}}(\zeta) \exp\left(\frac{\hbar}{2i} \Theta^{\eta\zeta}\right) (X(\eta)Y(\zeta) - (-1)^{\varepsilon_X \varepsilon_Y} Y(\eta)X(\zeta)) \right) \Big|_{\eta=\zeta},\end{aligned}\tag{3.9a}$$

$$\begin{aligned}\hat{\mathcal{P}}\{X, Y\} &= \frac{1}{2}(\hat{\mathcal{P}}(XY) + (-1)^{\varepsilon_X \varepsilon_Y} \hat{\mathcal{P}}(YX)) \\ &= \left( \hat{\mathcal{P}}(\eta) \hat{\mathcal{P}}(\zeta) \exp\left(\frac{\hbar}{2i} \Theta^{\eta\zeta}\right) \frac{1}{2} (X(\eta)Y(\zeta) + (-1)^{\varepsilon_X \varepsilon_Y} Y(\eta)X(\zeta)) \right) \Big|_{\eta=\zeta}.\end{aligned}\tag{3.9b}$$

### 3.3 Star representation of Projected operator Algebra

The commutator formulas (3.7a), (3.9a) and the symmetrized product ones, (3.7b), (3.9b) can be reformulated in terms of the so-called star-product like the Moyal star-product. For this purpose, we introduce the operator-products in the quantum constraint systems as follows:

$$X \star Y = \exp\left(\frac{\hbar}{2i} \Theta^{\eta\zeta}\right) X(\eta)Y(\zeta) \Big|_{\eta=\zeta},\tag{3.10}$$

which we shall call the constraint  $\star$ -product, and

$$X \hat{\mathcal{P}} \star Y = \left( \hat{\mathcal{P}}(\eta) \hat{\mathcal{P}}(\zeta) \exp\left(\frac{\hbar}{2i} \Theta^{\eta\zeta}\right) X(\eta)Y(\zeta) \right) \Big|_{\eta=\zeta},\tag{3.11}$$

which, the constraint  $\hat{\mathcal{P}}\star$ -product.

We next define two kinds of  $\star$ -commutators and  $\hat{\mathcal{P}}\star$ -symmetrized products as follows:

$$\begin{aligned}[X, Y]_\star &= X \star Y - (-1)^{\varepsilon_X \varepsilon_Y} Y \star X \quad (\star\text{-commutator}), \\ \{X, Y\}_\star &= \frac{1}{2}(X \star Y + (-1)^{\varepsilon_X \varepsilon_Y} Y \star X) \quad (\star\text{-symmetrized product})\end{aligned}\tag{3.12}$$

and

$$\begin{aligned}[X, Y]_{\hat{\mathcal{P}}\star} &= X \hat{\mathcal{P}} \star Y - (-1)^{\varepsilon_X \varepsilon_Y} Y \hat{\mathcal{P}} \star X \quad (\hat{\mathcal{P}}\star\text{-commutator}), \\ \{X, Y\}_{\hat{\mathcal{P}}\star} &= \frac{1}{2}(X \hat{\mathcal{P}} \star Y + (-1)^{\varepsilon_X \varepsilon_Y} Y \hat{\mathcal{P}} \star X) \quad (\hat{\mathcal{P}}\star\text{-symmetrized product}).\end{aligned}\tag{3.13}$$

Finally, the nonlocal representations of the commutators and the symmetrized products (3.3) and (3.4) are reformulated by using the constraint  $\star$ ,  $\hat{\mathcal{P}}\star$ -product formulation as follows:

$$\begin{aligned} [\hat{\mathcal{P}}X, \hat{\mathcal{P}}Y] &= (-1)^s \hat{\mathcal{P}}[X, Y]_\star \\ \{\hat{\mathcal{P}}X, \hat{\mathcal{P}}Y\} &= \frac{1}{2} \hat{\mathcal{P}}\{X, Y\}_\star \end{aligned} \quad (3.14)$$

and

$$\begin{aligned} \hat{\mathcal{P}}[X, Y] &= -(-1)^s [X, Y]_{\hat{\mathcal{P}}\star} \\ \hat{\mathcal{P}}\{X, Y\} &= \{X, Y\}_{\hat{\mathcal{P}}\star}. \end{aligned} \quad (3.15)$$

Successively applying the formulas (3.14) and (3.15), the commutator of the projected operators  $\hat{\mathcal{P}}X$  and  $\hat{\mathcal{P}}Y$  can be represented in the form of the power series of  $\hbar$  as follows:

$$[\hat{\mathcal{P}}X, \hat{\mathcal{P}}Y] = \sum_{n=0}^{\infty} \hbar^n C_n(\mathcal{C}^*), \quad (3.16)$$

where the first two terms are the quantized form of the Dirac bracket and the rest higher order terms of  $\hbar$  are the quantum corrections caused by the noncommutativity among the ACCS and the operators  $X, Y$ . Similarly, the projection of the symmetrized product,  $\hat{\mathcal{P}}\{X, Y\}$  is represented in the following form:

$$\hat{\mathcal{P}}\{X, Y\} = \sum_{n=0}^{\infty} \hbar^{2n} S_n(\mathcal{C}^*), \quad (3.17)$$

which contains the quantum corrections in the form of the power series of  $\hbar^2$ .

## 4 Conclusions

We have proposed the new types of star-products in the quantization of the second-class constraint systems with the operator formalism. Although the ordinary star-product

$$f(z) \star g(z) = \exp\left(\frac{i\hbar}{2} \theta^{\mu\nu} \frac{\partial}{\partial z_1^\mu} \frac{\partial}{\partial z_2^\nu}\right) f(z_1^\mu) g(z_2^\nu) \Big|_{z_1=z_2=z}, \quad (4.1)$$

is expressed in terms of the original CCS,  $\{z^\mu = (q^i, p_i : i = 1, \dots, N)\}$ , with the nonlocal representation, the star-products (3.10) and (3.11) are described in terms of the *hyper*-operators  $\hat{Z}_\alpha^{(-)}$  ( $\alpha = 1, \dots, 2M$ ) of the ACCS, which is the subset the modified CCS,  $\mathcal{C}^* \oplus \{Z_\alpha\}$ . Then, one sees that the new star-products defined by

(3.10) and (3.11) are the projections of the ordinary star-products to  $\mathcal{C}^*$  in the operator formalism.

By using these star-products, we have derived the commutator formulas and the symmetrized product ones, (3.14) and (3.15), in the projected quantum system. Successively applying these formulas, it has been shown that the commutators among the projected operators and the operator products contain the quantum effect caused by the noncommutativity among the ACCS and the operators in the form of the power series of  $\hbar$ .

## Appendix A Some Projection Formulas

In order to calculate the projection of such a type of  $X(\hat{Z}^{(+)})^n Y$ , here, the following formulas are presented.

$$\hat{\mathcal{P}}[X, \hat{Z}_{\alpha_1}^{(+)} \dots \hat{Z}_{\alpha_{2n}}^{(+)} Y] = (-1)^{ns} \left( \frac{\hbar}{2i} \right)^{2n} \hat{\mathcal{P}}[\hat{Z}_{\alpha_{2n}}^{(-)} \dots \hat{Z}_{\alpha_1}^{(-)} X, Y] \quad (A1)$$

$$\hat{\mathcal{P}}[X, \hat{Z}_{\alpha_1}^{(+)} \dots \hat{Z}_{\alpha_{2n+1}}^{(+)} Y] = 2(-1)^{\varepsilon_X s + ns} \left( \frac{\hbar}{2i} \right)^{2n+1} \hat{\mathcal{P}}\{\hat{Z}_{\alpha_{2n+1}}^{(-)} \dots \hat{Z}_{\alpha_1}^{(-)} X, Y\} \quad (A2)$$

$$\hat{\mathcal{P}}\{X, \hat{Z}_{\alpha_1}^{(+)} \dots \hat{Z}_{\alpha_{2n}}^{(+)} Y\} = (-1)^{ns} \left( \frac{\hbar}{2i} \right)^{2n} \hat{\mathcal{P}}\{\hat{Z}_{\alpha_{2n}}^{(-)} \dots \hat{Z}_{\alpha_1}^{(-)} X, Y\} \quad (A3)$$

$$\hat{\mathcal{P}}\{X, \hat{Z}_{\alpha_1}^{(+)} \dots \hat{Z}_{\alpha_{2n+1}}^{(+)} Y\} = \frac{1}{2}(-1)^{\varepsilon_X s + ns} \left( \frac{\hbar}{2i} \right)^{2n+1} \hat{\mathcal{P}}[\hat{Z}_{\alpha_{2n+1}}^{(-)} \dots \hat{Z}_{\alpha_1}^{(-)} X, Y] \quad (A4)$$

## References

- [1] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems* (Princeton University Press, Princeton (1992)).
- [2] P.A.M. Dirac, *Lectures on Quantum Mechanics* (Belfer Graduate School of Science, Yeshiva University, New York) 1969.
- [3] I.A. Batalin and E.S. Fradkin, Nucl. Phys. B **279** (1987) 514.

- [4] R.Loll, Phys.Rev.D **41** (1990) 3785;  
 G.Dune, R.Jackiw and C.A.Trugenberger, Ann.Phys.(N.Y.) **194** (1989);  
 E.Witten, Commun.Math.Phys. **121** (1989) 351;  
 G.Jorjadze, J.Math.Phys. **36** (1997) 851.
- [5] M.Nakamura and N.Mishima, Prog.Theor.Phys. **81** (1989) 514;  
 M.Nakamura and H.Minowa, J.Math.Phys. **34** (1993) 50.
- [6] M.L.Krivoruchenko,A.A.Raduta and A.Faessler, arXiv:hep-th/0507049v1.
- [7] Y.Ohnuki and S.Kitakado, J.Math.Phys. **34** (1993) 2827.
- [8] H.Weyl, *The Theory of Groups and Quantum Mechanics*, Dover Publications,  
 New York Inc.) 1931;  
 E.P.Wigner, Phys.Rev. **40** (1932) 749.
- [9] J.E.Moyal, Proc.Cambridge Phil.Soc. **45** (1949) 99.
- [10] M.I.Krivoruchenko, arXiv.:hep-th/0610074.
- [11] T.Hori, T.Koikawa and T.Maki, arXiv.:hep-th/0206190.
- [12] H.Groenewold, Physica **12** (1946) 405.